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STATIONARY SOLUTION OF THE EQUATIONS OF MICROCONVECTION IN A VERTICAL LAYER

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The stationary problem of convection in liquids is considered using the model of microconvection developed by V. V. Pukhnachev. Velocity profiles for boundary conditions of different classes are constructed. The solutions of the problem under study and the classical problem based on the Oberbeck–Boussinesq model are compared.

System of the Equations of Microconvection. Pukhnachev [1] developed a model of microconvection in which the temperature dependence of density has the form

$$\rho = \rho_0(1 + \beta\Theta)^{-1} \tag{1}$$

(ρ_0 and β are the positive constants). The mathematical formulation of the model is as follows:

$$\operatorname{div} \boldsymbol{w} = 0; \tag{2}$$

$$\boldsymbol{w}_t + \boldsymbol{w} \cdot \nabla \boldsymbol{w} + \beta\chi(\nabla\Theta \cdot \nabla \boldsymbol{w} - \nabla \boldsymbol{w} \cdot \nabla\Theta) + \beta^2\chi^2(\Delta\Theta\nabla\Theta - \nabla|\nabla\Theta|^2/2) = (1 + \beta\Theta)(-\nabla q + \nu\Delta \boldsymbol{w}) + \boldsymbol{g}; \tag{3}$$

$$\Theta_t + \boldsymbol{w} \cdot \nabla\Theta + \beta\chi|\nabla\Theta|^2 = (1 + \beta\Theta)\chi\Delta\Theta, \tag{4}$$

where $\boldsymbol{w} = \boldsymbol{v} - \beta\chi\nabla\Theta$ and $q = \rho_0^{-1}(p - \lambda \operatorname{div} \boldsymbol{v}) - \beta(\nu - \chi)\chi\Delta\Theta$ are unknown functions [$\boldsymbol{v} = (v_1, v_2, v_3)$ is the true velocity vector], β is the coefficient of thermal expansion, $\chi = k/(\rho_0 c)$ is the thermal diffusivity (the thermal conductivity k and the specific heat of the liquid c are constant), p is the true liquid pressure, λ is the second viscosity coefficient, $\nu = \mu/\rho_0$ is the kinematic viscosity, and μ is the dynamic viscosity.

The functions $\boldsymbol{w}(\boldsymbol{x}, t)$, $q(\boldsymbol{x}, t)$, and $\Theta(\boldsymbol{x}, t)$ are solutions of system (2)–(4) with the following boundary conditions on the rigid walls:

$$\boldsymbol{w} + \beta\chi\nabla\Theta = 0, \qquad \Theta = \Theta_w(\boldsymbol{x}, t) \tag{5}$$

or

$$\boldsymbol{w} + \beta\chi\nabla\Theta = 0, \qquad \frac{\partial\Theta}{\partial n} + \delta(\Theta - \Theta_{\text{amb}}) = d. \tag{6}$$

Here Θ_{amb} is the ambient temperature. The first condition in (5) is the attachment condition ($\boldsymbol{v} = 0$) on the immovable rigid wall and the second condition specifies the wall temperature. The second condition in (6) characterizes heat exchange with the ambient medium (heat flux is specified at $\delta = 0$). In addition, for nonstationary motions it is necessary to specify the initial conditions $\boldsymbol{w} = \boldsymbol{w}_0(\boldsymbol{x})$, $\operatorname{div} \boldsymbol{w}_0 = 0$, and $\Theta = \Theta_0(\boldsymbol{x})$ at $t = 0$.

Remark 1. Pukhnachev [1] derived system (2)–(4) for determining the functions $\boldsymbol{w}$, q, and Θ from the exact equations of continuity, momentum, and energy. In [2], he showed that the classical Oberbeck–Boussinesq approximation is inadequate for describing thermal gravitational convection if the dimensionless parameter $\varepsilon_1 = |g|a^3/(\nu\chi)$ (a is the characteristic linear dimension) is smaller than or equal to unity. For a given liquid, a small value of the parameter ε_1 can be ensured by the smallness of the gravitational acceleration $\boldsymbol{g}$ or the length scale a. The solvability of problem (2)–(4) in the Hölder classes is established in [3].

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Remark 2. The momentum equation (3) can be written in different form. Indeed, $\nabla\Theta \cdot \nabla \boldsymbol{w} - \nabla \boldsymbol{w} \cdot \nabla\Theta = \nabla\Theta \cdot \nabla \boldsymbol{w} - \nabla\Theta \cdot (\nabla w)^* = \operatorname{rot} \boldsymbol{w} \times \nabla\Theta$ (asterisk denotes conjugation). Next, $\nabla|\nabla\Theta|^2/2 = \nabla\Theta \cdot \nabla(\nabla\Theta)$, and, hence, $\Delta\Theta\nabla\Theta - \nabla|\nabla\Theta|^2/2 = [\operatorname{div}(\nabla\Theta)I - \partial(\nabla\Theta)/\partial \boldsymbol{x}]\nabla\Theta = \operatorname{div}[\nabla\Theta \otimes \nabla\Theta - |\nabla\Theta|^2 I]$, where I is a unit tensor and the symbol "$\otimes$" denotes a tensor product. With allowance for the above formulas, Eq. (3) is written as

$$\boldsymbol{w}_t + \boldsymbol{w} \cdot \nabla \boldsymbol{w} + \beta\chi \operatorname{rot} \boldsymbol{w} \times \nabla\Theta + \beta^2\chi^2 \operatorname{div}[\nabla\Theta \otimes \nabla\Theta - |\nabla\Theta|^2 I] = (1 + \beta\Theta)(-\nabla q + \nu\Delta \boldsymbol{w}) + \boldsymbol{g},$$

which is more convenient for *a priori* estimates.

Solution of the Stationary Problem in the Case of a Special Temperature Distribution. Let us choose a coordinate system so that $\boldsymbol{g} = (0, -g, 0)$ and assume that the liquid occupies the layer $|x| < a$ and the boundaries of the layer are solid surfaces with specified heat flux. If the value of the heat flux does not depend on z, plane flows are possible in a vertical layer. They can develop if the initial velocity and temperature distributions do not depend on z or the velocity component $v_3 = 0$ at $t = 0$. Below, we consider only steady flows in the layer.

In the plane case, system (2)–(4) for steady flow ($\boldsymbol{w}_t = 0$ and $\Theta_t = 0$) admits the operators $\partial/\partial y$ and $\psi\partial/\partial q$, and this reflects its invariance with respect to transposition along the y axis and increase in q (analog of pressure) by an arbitrary constant ψ. The invariant solutions of system (2)–(4) with respect to the operator $\partial/\partial y + \psi\partial/\partial q$ are written as

$$\boldsymbol{w} = (w_1, w_2, 0), \quad w_1(x) \equiv u, \quad w_2(x) \equiv v, \quad \Theta = \Theta(x), \quad q = (\varphi - g)y + r(x), \tag{7}$$

where $\varphi = \psi + g$. In the equation for q, the term $-gy$ corresponds to the hydrostatic constituent in the representation of the true pressure p. Substitution of (7) into system (2)–(4) causes disintegration of the system into several equations, which are solved subsequently with respect to the unknown functions $u(x)$, $v(x)$, $\Theta(x)$, and $r(x)$ ($\varphi = \text{const}$).

From the continuity equation (2) it follows that $w_1 = \text{const}$ and $w_2(x)$ is an arbitrary function. We assume that $u \equiv w_1 = u_0 = \text{const}$ and $v \equiv w_2$ is an arbitrary function.

With allowance for (7), the energy equation (4) takes the form $(u_0 + \beta\chi\Theta_x)\Theta_x = (1 + \beta\Theta)\chi\Theta_{xx}$. The last second-order equation has a two-parameter family of stationary solutions

$$\Theta(x) = \frac{1}{\beta}\Big[\frac{1}{c_1} - 1 + c_2 \exp\Big(\frac{c_1 u_0 x}{\chi}\Big)\Big] \qquad (c_1 \neq 0) \tag{8}$$

and a singular solution

$$\Theta_0(x) = \bar{\Theta} - u_0 x/(\beta\chi) \qquad (\bar{\Theta} = \text{const}). \tag{9}$$

According to (6), at $\delta = 0$ the boundary condition on the walls ($x = \pm a$) for solution (7) has the form $\Theta_x = -u_0/(\beta\chi) \equiv d$. The only parameter — the temperature field (9) — satisfies this condition at any constant $\bar{\Theta}$.

Projecting (3) onto the x axis, we obtain $(1 + \beta\Theta_0)(-r_x) = 0$, i.e., $r = r_0 = \text{const}$. The function $q(x, y)$ is determined with accuracy up to a constant, and, hence, we can assume that $r_0 \equiv 0$. Projection of (3) onto the y axis gives the equation

$$(u_0 + \beta\chi\Theta_{0x})v_x = (1 + \beta\Theta_0)(\nu v_{xx} - \varphi) + (1 + \beta\Theta_0)g - g = (1 + \beta\Theta_0)(\nu v_{xx} - \varphi) + \beta\Theta_0 g. \tag{10}$$

According to (9), $u_0 + \beta\chi\Theta_{0x} = 0$ and Eq. (10) reduces to

$$\Big(1 + \beta\bar{\Theta} - \frac{u_0 x}{\chi}\Big)(\nu v_{xx} - \varphi) + \beta\bar{\Theta}g - \frac{u_0 g}{\chi}x = 0. \tag{11}$$

Equation (11) is an ordinary differential equation of second order, in which φ is an unknown constant. Three conditions are required to solve this equation. Since $\nabla\Theta = (\Theta_{0x}, 0)$, we have

$$v(x) = 0, \qquad x = \pm a. \tag{12}$$

To define $v(x)$ uniquely, we should find the constant φ. Solution (7) describes approximately convection in the central part of a finite closed cavity whose length larger than width $2a$. The condition of zero mass flow of the liquid through any cross section of the layer $y = \text{const}$ is imposed on this solution:

$$\int\limits_{-a}^{a} \rho(x)v_2(x)\,dx = 0. \tag{13}$$

Here $v_2(x)$ is the true velocity and $\rho(x)$ is the density of the liquid. In solution (7), $v_2(x) = v(x)$. With allowance for the equation of state (1), we obtain

$$\int_{-a}^{a} \frac{v(x)}{1+\beta\Theta_0(x)}\, dx = 0. \tag{14}$$

The general solution of Eq. (11) is written as

$$v = \frac{1}{\nu}\Big[(\varphi - g)\,\frac{x^2}{2} + c_1 x + c_2 + \frac{g\chi^2}{u_0^2}\Big(1+\beta\bar{\Theta} - \frac{u_0 x}{\chi}\Big)\Big(\ln\Big(1+\beta\bar{\Theta} - \frac{u_0 x}{\chi}\Big) - 1\Big)\Big] \tag{15}$$

with the arbitrary constants c_1 and c_2 determined from (12):

$$\begin{aligned} c_1 &= -\frac{g\chi^2}{2au_0^2}\Big(f_1 \ln f_1 - f_2 \ln f_2 + 2\,\frac{u_0 a}{\chi}\Big), \\ c_2 &= \frac{g-\varphi}{2}\,a^2 - \frac{g\chi^2}{2u_0^2}\,(f_1 \ln f_1 + f_2 \ln f_2 - 2(1+\beta\bar{\Theta})). \end{aligned} \tag{16}$$

Here $f_1 = 1 + \beta\bar{\Theta} - u_0 a/\chi$ and $f_2 = 1 + \beta\bar{\Theta} + u_0 a/\chi$.

Substituting (15) into (14) and taking into account (16), we find

$$\varphi = g\Big\{1 - \Big[\frac{\chi}{2u_0 a}\,(\ln f_2 - \ln f_1)\Big(f_1 \ln f_1 + f_2 \ln f_2 + \frac{\chi}{u_0 a}\,(1+\beta\bar{\Theta})(f_1 \ln f_1 - f_2 \ln f_2)\Big) + 2\Big]$$
$$\times \Big[1 + \beta\bar{\Theta} + (\ln f_2 - \ln f_1)\Big(\frac{u_0 a}{2\chi} - \frac{\chi}{2u_0 a}\,(1+\beta\bar{\Theta})^2\Big)\Big]^{-1}\Big\}.$$

Let us compare the solution obtained to the solution of the stationary problem in the Oberbeck–Boussinesq approximation. The temperature fields in both solutions coincide; moreover, the horizontal velocity component is equal to zero in both the classical approximation and the new solution. However, their vertical components differ: in the Oberbeck–Boussinesq model, this component has the form

$$v = -\frac{gu_0}{6\nu\chi}\,x(a^2 - x^2). \tag{17}$$

To compare formulas (15) and (17), we write them in dimensionless form. For this, we choose the characteristic linear dimension $x = \eta a$ and introduce the nondimensional parameters $\gamma = u_0 a/\chi$ and $\varepsilon = \beta\bar{\Theta}$. Converting to the nondimensional variable, we obtain

$$v = -\frac{gu_0}{6\nu\chi}\,a^3\eta(1-\eta^2).$$

Here the coefficient $gu_0 a^3/(6\nu\chi)$ is in centimeters per second. Dividing v by this coefficient and denoting the resulting expression by v_b, we finally have

$$v_b = -\frac{6\nu\chi}{gu_0 a^3}v = \eta - \eta^3. \tag{18}$$

In the new model, we denote the velocity by v_n. Hence, in accordance with (15), we have

$$v_n = -\frac{\nu}{ga^2}\,v = \frac{p_1(\varepsilon,\gamma)}{4\gamma p_2(\varepsilon,\gamma)}\,\eta^2 - \frac{p_3(\varepsilon,\gamma)}{2\gamma^2}\,\eta + \frac{p_1(\varepsilon,\gamma)}{4\gamma p_2(\varepsilon,\gamma)} - \frac{p_4(\varepsilon,\gamma)}{2\gamma^2} + \frac{1+\varepsilon-\gamma\eta}{\gamma^2}\,(\ln(1+\varepsilon-\gamma\eta) - 1), \tag{19}$$

where

$$p_1(\varepsilon,\gamma) = (\ln f_2^* - \ln f_1^*)\Big[f_1^* \ln f_1^* + f_2^* \ln f_2^* + \frac{1+\varepsilon}{\gamma}(f_1^* \ln f_1^* - f_2^* \ln f_2^*)\Big] + 4\gamma;$$

$$p_2(\varepsilon,\gamma) = 1 + \varepsilon + (\ln f_2^* - \ln f_1^*)\Big(\frac{\gamma}{2} - \frac{1}{2\gamma}\,(1+\varepsilon)^2\Big);$$

$$p_3(\varepsilon,\gamma) = f_1^* \ln f_1^* - f_2^* \ln f_2^* + 2\gamma; \quad p_4(\varepsilon,\gamma) = f_1^* \ln f_1^* + f_2^* \ln f_2^* - 2(1+\varepsilon).$$

The constants $f_1^* = 1 + \varepsilon - \gamma$ and $f_2^* = 1 + \varepsilon + \gamma$ correspond to f_1 and f_2, respectively.

Comparison of Eq. (18) and Eq. (19) shows that the function v_n is not an uneven function, in contrast to the vertical velocity distribution for a steady stratified convective flow in a vertical layer, which is an odd function, according to the classical model of convection. Velocity profiles are shown in Fig. 1, where curve 1 is a profile of the velocity $v_b = v$ in the classical model and curves 2–5 are profiles of the velocity $v_n = v \cdot 10^2$ for the new model

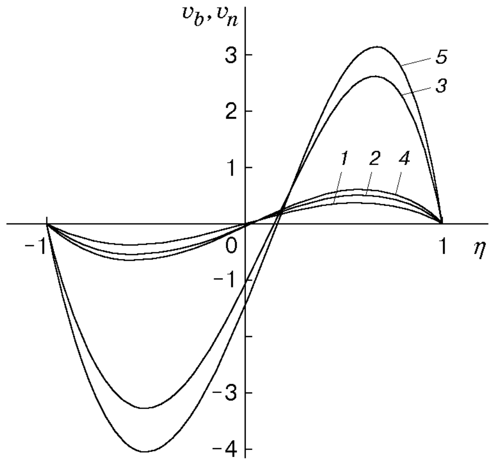

Fig. 1

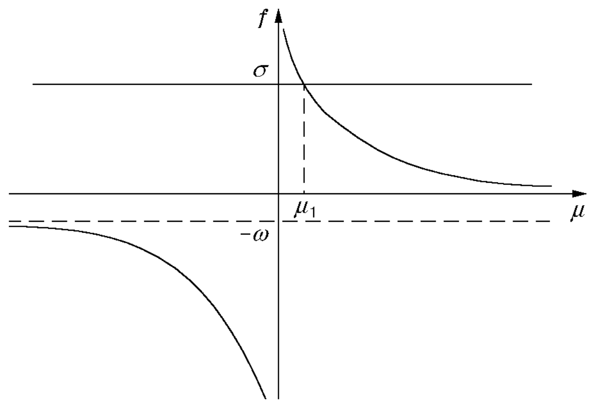

Fig. 2

[$\varepsilon = 0.1$ and $\gamma = 0.1$ (2), $\varepsilon = 0.1$ and $\gamma = 0.5$ (3), $\varepsilon = 0.01$ and $\gamma = 0.1$ (4), and $\varepsilon = 0.01$ and $\gamma = 0.5$ (5)]. It is evident that with increase in wall temperature difference, the value of $v_n(0)$ grows, and the maximum value $v_n^{\max}$ is shifted toward the heated wall.

Remark 3. Pukhnachev [2] studied motion in a vertical slit [for solution (9) and its nonstationary variant]; boundary conditions were taken in the form (6). However, in [2], some inaccuracies are made in the formulas of temperature, and, hence, velocity, and explicit formulas of velocity necessary for stability analysis of this flow are lacking.

Analysis of the Stationary Solution at Specified Wall Temperature. Let us consider the case where boundary conditions are taken in the form (5), i.e., the temperatures θ_1 and θ_2 are specified on the walls $x = \pm a$. According to the second conditions of (5), we have $\Theta|_{x=-a} = \theta_1$ and $\Theta|_{x=a} = \theta_2$. From (9), we obtain $\theta_1 = \bar\Theta + u_0 a/(\beta\chi)$ and $\theta_2 = \bar\Theta - u_0 a/(\beta\chi)$. Thus, the singular solution (9) satisfies these conditions if the constants u_0 and $\bar\Theta$ depend on θ_1 and θ_2 as follows:

$$\bar\Theta = (\theta_1 + \theta_2)/2, \qquad u_0 = (\theta_1 - \theta_2)\beta\chi/(2a).$$

In this case, the density is positive ($\beta\bar\Theta > -1$) and $\Theta > 0$ in the layer $|x| < a$.

For stationary solutions of the form (8), the following conditions must be satisfied on the walls:

$$\frac{1}{\beta}\Big[\frac{1}{c_1} - 1 + c_2 \exp\Big(-\frac{c_1 u_0 a}{\chi}\Big)\Big] = \theta_1, \qquad \frac{1}{\beta}\Big[\frac{1}{c_1} - 1 + c_2 \exp\Big(\frac{c_1 u_0 a}{\chi}\Big)\Big] = \theta_2. \tag{20}$$

Subtracting the first of Eqs. (20) from the second, we obtain

$$\frac{c_2}{\beta} = \frac{\theta_2 - \theta_1}{\exp(\mu) - \exp(-\mu)}, \tag{21}$$

where $\mu = c_1 u_0 a/\chi$.

We consider the following possible variants.

1. If $\theta_1 = \theta_2$, then $c_2 = 0$ and $c_1 = 1/(1 + \beta\theta_1)$, i.e., the layer temperature is constant.

2. Let $\theta_1 \neq \theta_2$. Substituting Eq. (21) into the first of Eqs. (20) and replacing $\omega = \beta\chi(\theta_2 - \theta_1)/(u_0 a)$ and $\sigma = (1 + \beta\theta_1)\chi/(u_0 a)$, we obtain the equation

$$\frac{1}{\mu} + \frac{\omega}{\exp(2\mu) - 1} = \sigma. \tag{22}$$

We note that $\sigma > 0$ (assuming $u_0 > 0$). Let us determine whether solutions of Eq. (22) exist. Let $f(\mu) = 1/\mu + \omega/(\exp(2\mu) - 1)$. Then, the derivative of this function has the form

$$f'(\mu) = -\frac{1}{\mu^2} - \frac{2\omega\exp(2\mu)}{(\exp(2\mu) - 1)^2}. \tag{23}$$

2′. At $\omega > 0$ ($\theta_2 > \theta_1$), the function $f(\mu)$ decreases monotonically [$f'(\mu) < 0$] in the domain of definition. Therefore, there is a unique solution $f(\mu_1) = \sigma$ and $\mu_1 > 0$ (Fig. 2), i.e., there are unique constants

$$c_1 = \frac{\chi}{u_0 a}\,\mu_1 > 0, \qquad c_2 = \frac{2\beta(\theta_2 - \theta_1)}{\sinh\mu_1}. \tag{24}$$

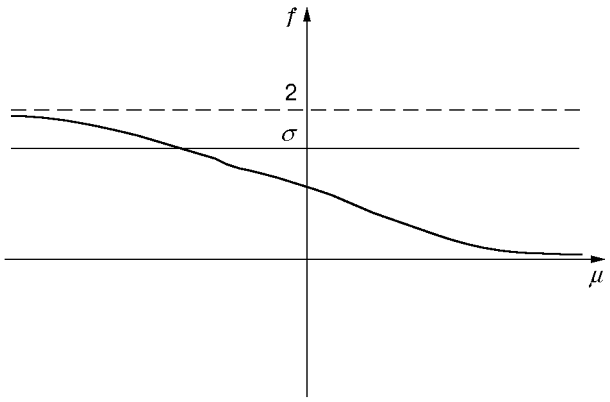

Fig. 3

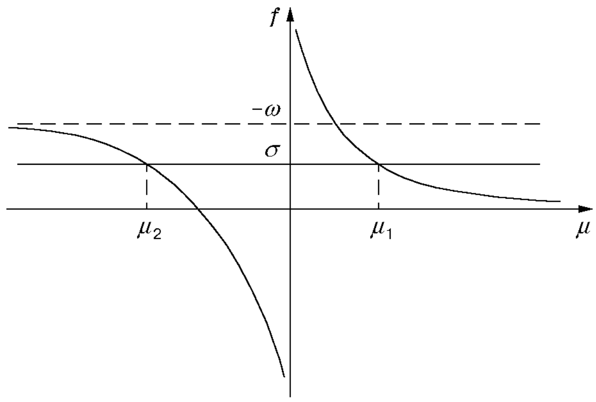

Fig. 4

Let $\omega < 0$. For $\mu \to 0$, we have $f(\mu) \approx (1/\mu)[1+\omega/2-\omega\mu/2]$, and, hence, the cases $1+\omega/2 = 0$, $-2 < \omega < 0$, and $\omega < -2$ are to be considered.

3. Let $1 + \omega/2 = 0$, i.e., $\omega = -2$. Then, $f(+\infty) = 0$, $f(-\infty) = 2$, and $f \to 1$ as $\mu \to \pm 0$. According to (23), we have derivative $f'(\mu) = -1/\mu^2 + 4\exp(2\mu)/(\exp(2\mu) - 1)^2$ at $\omega = -2$. Let us determine whether $f(\mu)$ has points of local extremum. It can be shown that $f'(\mu_*) = 0$ if and only if the equality $|\sinh \mu_*| = |\mu_*|$ is satisfied. The latter can be satisfied only for $\mu_* = 0$, but in this case, $f(0) = 1$ at the point $\mu = 0$. The derivative $f'(\mu) \leqslant 0$ $[f'(\mu) = 0$ only for $\mu = 0]$, and hence, the function $f(\mu)$ decreases over the entire axis. Therefore, for $\omega = -2$, the unique solution of Eq. (22) is μ_1, and it exists only for $0 < \sigma < 2$. This means that the uniquely determined constants c_1 and c_2 exist for the specified values of σ (Fig. 3).

4. Let $1 + \omega/2 > 0$, i.e., $\omega > -2$. Then, $f(\mu) \sim (1 + \omega/2)/\mu$. The function $f(\mu)$ has a discontinuity at the point $\mu = 0$ and $f(\pm 0) = \pm\infty$. If $f'(\mu_*) = 0$, the following cases are possible:

(a) $-2 < \omega < 0$;

(b) $\omega > 0$ (this case reduces to the variant $2'$).

In the case (a), we have $-\omega = |\omega|$. We consider the equation $f'(\mu) = 0$, i.e.,

$$f'(\mu) = -\frac{1}{\mu^2} + \frac{|\omega|}{2\sinh^2\mu} = 0. \tag{25}$$

The derivative is equal to zero only for $\sinh^2\mu_* = |\omega|\mu_*^2/2$ $(\mu_* \neq 0)$. We note that if μ is a solution of Eq. (25), $-\mu$ is also a solution of (25). Hence, we assume that $\mu_* > 0$. Then, $\sinh\mu_* = \sqrt{|\omega|/2}\mu_*$. It is easy to show that Eq. (25) has a solution only for $|\omega| > 2$, and since in this case, $-2 < \omega < 0$, the last equation has no solutions. Thus, the derivative $f'(\mu)$ conserves sign in the domain $[f'(\mu) < 0]$, and the function $f(\mu)$ decreases.

If $0 < \sigma < -\omega$, Eq. (22) has two solutions: $\mu_1 > 0$ and $\mu_2 < 0$. Thus, there are two pairs of constants: (c_1^1, c_2^1) and (c_1^2, c_2^2). If $\sigma \geqslant -\omega$, Eq. (22) has one solution (Fig. 4).

5. Let $1+\omega/2 < 0$, i.e., $\omega < -2$ and $-\omega = |\omega|$. The function $f(\mu) \sim (1+\omega/2)/\mu$, while for $\mu \to 0$, according to the definition of the function, $f(+0) \to -\infty$ and $f(-0) \to +\infty$. It is easy to show that solutions of the equation

$$f'(\mu) = \frac{1}{\mu^2\sinh^2\mu}\Big[\frac{|\omega|}{2}\mu^2 - \sinh^2\mu\Big] = 0$$

are $\mu_* > 0$ and $-\mu_*$ if the inequality $|\omega| > 2$ is satisfied. Thus, $f(-\mu_*) = -1/\mu_* - |\omega|(\exp(-2\mu_*) - 1)$ and $f(\mu_*) = \omega(\exp(2\mu_*)-1)+1/\mu_*$. Generally speaking, for $f(-\mu_*) < f(\mu_*)$, Eq. (22) has four solutions if $f(-\mu_*) < \sigma < f(\mu_*)$. Let us show that $f(\mu_*) < f(-\mu_*)$. In other words, we need to show that $2/\mu_* < |\omega|\coth\mu_*$. The last inequality is equivalent to $\tanh\mu_* < |\omega|\mu_*/2$ $(\mu_* > 0)$. To verify this inequality, we consider the function $h(\mu) = \tanh\mu - |\omega|\mu/2$. We have $h(0) = 0$ and $h'(\mu) = 1/\cosh^2\mu - |\omega|/2$. Let us determine the sign of the derivative at the point $\mu = 0$. It is obvious that $h'(0) = 1 - |\omega|/2 < 0$. Since $\cosh\mu > 1$, the condition $h'(\mu) < 0$ is satisfied, i.e., the function $h(\mu)$ decreases. Hence, $h(\mu) < 0$ for $\mu > 0$, i.e., $\tanh\mu_* < |\omega|\mu_*/2$ or $f(\mu_*) < f(-\mu_*)$. Thus, if $f(\mu_*) < \sigma < f(-\mu_*)$, Eq. (22) is not solvable. If $f(-\mu_*) < \sigma < -\omega$ or $0 < \sigma < f(\mu_*)$, Eq. (22) has two solutions, and if $\sigma = f(\mu_*)$, $\sigma = f(-\mu_*)$, or $\sigma \geqslant -\omega$, it has a unique solution (Fig. 5).

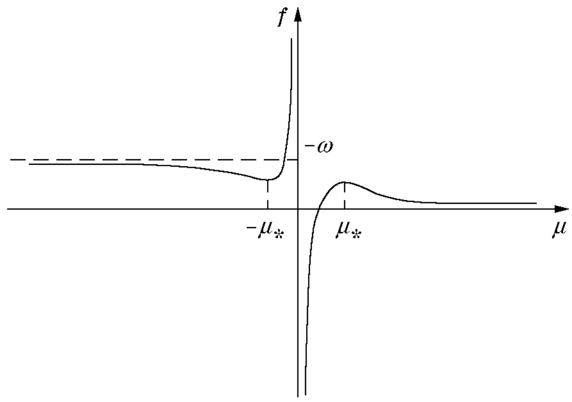

Fig. 5

We now verify that the conditions leading to uniqueness of the solution of Eq. (22) are satisfied. For all cases described above, the inequality $\sigma < -\omega$ must be valid

$$(1+\beta\theta_1)\,\frac{\chi}{u_0 a} < -\frac{\beta\chi(\theta_2-\theta_1)}{u_0 a},$$

whence we obtain $\theta_2 < 0$. Thus, none of the conditions under which Eq. (22) can have two solutions is satisfied. Consequently, for any values of θ_1 and θ_2, Eq. (22) has a unique solution. The unique values of c_1 and c_2 are determined from (24).

Taking $\Theta(x)$ in the form (8) with the specified c_1 and c_2, we project (3) onto the y axis. As a result, we obtain the following equation for determining the vertical component of the velocity v:

$$(1/c_1 + c_2\exp(kx))\nu v_{xx} - c_1c_2u_0\exp(kx)v_x = f(x), \tag{26}$$

where $k = c_1u_0/\chi$ and $f(x) = \varphi(1/c_1 + c_2\exp(kx)) - g(1/c_1 - 1 + c_2\exp(kx))$.

After substitution $z = 1 + c_1c_2\exp(kx)$, the general solution of Eq. (26) has the form

$$v = \int\limits_{h_1}^{z} \frac{z^\alpha}{z-1}\Big[\frac{\varphi-g}{\nu k^2}\int\frac{dz}{z^\alpha(z-1)} + \frac{gc_1}{\nu k^2}\int\frac{dz}{z^{\alpha+1}(z-1)} + D_1\Big]dz + D_2, \tag{27}$$

where D_1 and D_2 are constants, $h_1 = 1 + c_1c_2\exp(-ak)$, and $\alpha = \chi/\nu \equiv 1/\mathrm{Pr}$ (Pr is the Prandtl number). The constant φ is determined from (13):

$$\varphi = g(1 - c_1F_1) - D_1\nu k^2F_2.$$

Here

$$F_1 = \int\limits_{h_1}^{h_2}\frac{1}{z}\int\limits_{h_1}^{z}\frac{\sigma^\alpha}{\sigma-1}\int\limits_{h_1}^{\sigma}\frac{d\tau}{\tau^{\alpha+1}(\tau-1)}\,d\sigma\,dz\left(\int\limits_{h_1}^{h_2}\frac{1}{z}\int\limits_{h_1}^{z}\frac{\sigma^\alpha}{\sigma-1}\int\limits_{h_1}^{\sigma}\frac{d\tau}{\tau^{\alpha}(\tau-1)}\,d\sigma\,dz\right)^{-1};$$

$$F_2 = \int\limits_{h_1}^{h_2}\frac{1}{z}\int\limits_{h_1}^{z}\frac{\sigma^\alpha}{\sigma-1}\,d\sigma\,dz\left(\int\limits_{h_1}^{h_2}\frac{1}{z}\int\limits_{h_1}^{z}\frac{\sigma^\alpha}{\sigma-1}\int\limits_{h_1}^{\sigma}\frac{d\tau}{\tau^{\alpha}(\tau-1)}\,d\sigma\,dz\right)^{-1}.$$

The constants of integration D_1 and D_2 are determined from the condition of attachment on a stationary solid wall:

$$D_1 = \left(\frac{gc_1F_1}{\nu k^2}\int\limits_{h_1}^{h_2}\frac{z^\alpha}{z-1}\int\limits_{h_1}^{z}\frac{d\sigma}{\sigma^\alpha(\sigma-1)}\,dz - \frac{gc_1}{\nu k^2}\int\limits_{h_1}^{h_2}\frac{z^\alpha}{z-1}\int\limits_{h_1}^{z}\frac{d\sigma}{\sigma^{\alpha+1}(\sigma-1)}\,dz\right)$$

$$\times\left(\int\limits_{h_1}^{h_2}\frac{z^\alpha}{z-1}\,dz - F_2\int\limits_{h_1}^{h_2}\frac{z^\alpha}{z-1}\int\limits_{h_1}^{z}\frac{d\sigma}{\sigma^\alpha(\sigma-1)}\,dz\right)^{-1}, \qquad D_2 = 0.$$

Here $h_2 = 1 + c_1c_2\exp(ak)$.

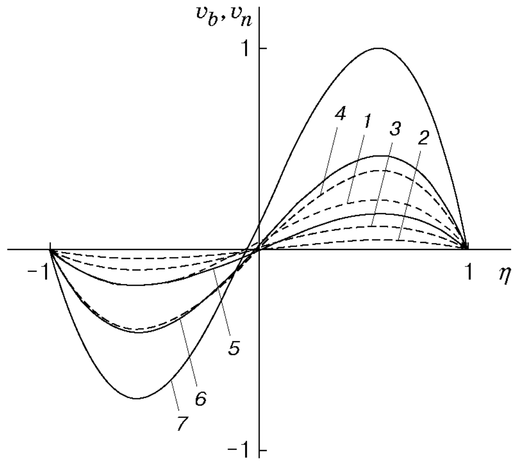

Fig. 6

To compare the solution obtained for the vertical component with the stationary problem in the classical formulation, we introduce the dimensionless parameter $\gamma_1 = ak$ and, substituting $z = 1 + c_1 c_2 \exp(\gamma_1 \eta)$, write Eq. (27) in dimensionless form

$$v_n = \frac{\nu k^2}{g} v(\eta) \qquad (-1 \leqslant \eta \leqslant 1), \tag{28}$$

where

$$v(\eta) = -(c_1 F_1 + D_1^* F_2)\gamma_1^2 \int_{-1}^{\eta} (1 + c_1 c_2 \exp(\gamma_1 \eta))^\alpha \int \frac{d\sigma}{(1 + c_1 c_2 \exp(\gamma_1 \sigma))^\alpha}\, d\eta$$

$$+ c_1 \gamma_1^2 \int_{-1}^{\eta} (1 + c_1 c_2 \exp(\gamma_1 \eta))^\alpha \int \frac{d\sigma}{(1 + c_1 c_2 \exp(\gamma_1 \sigma))^{\alpha+1}}\, d\eta + D_1^*;$$

$$D_1^* = \biggl(c_1 F_1 \gamma_1^2 \int_{-1}^{1} (1 + c_1 c_2 \exp(\gamma_1 \eta))^\alpha \int_{-1}^{\eta} \frac{d\sigma}{(1 + c_1 c_2 \exp(\gamma_1 \sigma))^\alpha}\, d\eta$$

$$- c_1 \gamma_1^2 \int_{-1}^{1} (1 + c_1 c_2 \exp(\gamma_1 \eta))^\alpha \int_{-1}^{\eta} \frac{d\sigma}{(1 + c_1 c_2 \exp(\gamma_1 \sigma))^{\alpha+1}}\, d\eta\biggr) \biggl(\gamma_1 \int_{-1}^{1} (1 + c_1 c_2 \exp(\gamma_1 \eta))^\alpha\, d\eta$$

$$- F_2 \gamma_1^2 \int_{-1}^{1} (1 + c_1 c_2 \exp(\gamma_1 \eta))^\alpha \int_{-1}^{\eta} \frac{d\sigma}{(1 + c_1 c_2 \exp(\gamma_1 \sigma))^\alpha}\, d\eta\biggr)^{-1}.$$

Velocity profiles are shown in Fig. 6, where curve 1 is a profile of the velocity $v_b = v$ in the classical model and the curves 2–7 are profiles of the velocity $v_n = v \cdot 10^3$ in the new model for the temperature differences $\Delta\Theta = 20$ (2), 50 (3), 100 (4), 20 (5), 50 (6), and 100°C (7); the solid and dashed curves refer to $a = 0.05$ and 0.025 cm, respectively. Evidently, the maximum value of $v_n^{\max}$ increases with increase in $\Delta\Theta$; moreover, profile (28) is characterized by a shift of the value of $v_n^{\max}$ toward the heated wall.

Calculations were performed for melted silicon at $u_0 = 1$. The values obtained are as follows:

1) for $a = 0.025$ cm and $\Delta\Theta = 20$°C,

$$c_1 = 9.8869319587 \cdot 10^{-1}, \qquad c_2 = 6.0661456314 \cdot 10^{-3},$$

$$\varphi = 1.6856451012 \cdot 10^{-1}, \qquad D_1 = -1.5264918983 \cdot 10^{-3};$$

2) for $a = 0.025$ cm and $\Delta\Theta = 50°$C,

$$c_1{=}9.9084990874 \cdot 10^{-1}, \qquad c_2{=}\ \ 1.5132327790 \cdot 10^{-2},$$

$$\varphi{=}2.3299026165 \cdot 10^{-1}, \qquad D_1{=}{-7.4848618079} \cdot 10^{-4};$$

3) for $a = 0.025$ cm and $\Delta\Theta = 100°$C,

$$c_1{=}9.9442304131 \cdot 10^{-1}, \qquad c_2{=}\ \ 3.0155820364 \cdot 10^{-2},$$

$$\varphi{=}3.3781464878 \cdot 10^{-1}, \qquad D_1{=}{-1.0261157689} \cdot 10^{-4};$$

4) for $a = 0.05$ cm and $\Delta\Theta = 20°$C,

$$c_1{=}9.8793651585 \cdot 10^{-1}, \qquad c_2{=}\ \ 3.0316983028 \cdot 10^{-3},$$

$$\varphi{=}1.4714983037 \cdot 10^{-1}, \qquad D_1{=}{-5.3249522658} \cdot 10^{-3};$$

5) for $a = 0.05$ cm and $\Delta\Theta = 50°$C,

$$c_1{=}9.8896436881 \cdot 10^{-1}, \qquad c_2{=}\ \ 7.5713428516 \cdot 10^{-3},$$

$$\varphi{=}1.8011127970 \cdot 10^{-1}, \qquad D_1{=}{-5.9644369079} \cdot 10^{-3};$$

6) for $a = 0.05$ cm and $\Delta\Theta = 100°$C,

$$c_1{=}9.9067235398 \cdot 10^{-1}, \qquad c_2{=}\ \ 1.5116493573 \cdot 10^{-2},$$

$$\varphi{=}2.3418316643 \cdot 10^{-1}, \qquad D_1{=}{-3.1642287370} \cdot 10^{-3}.$$

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